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THE DYNAMIC EVOLUTION OF A MECHANICAL SYSTEM WITH A VERY RIGID LINEAR DAMPER†

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A mechanical system consisting of two interacting subsystems is considered. When the interaction is removed one subsystem is Hamiltonian and the other is a dissipative linear oscillatory system. Integral manifolds are used to study the motions that are established after the high-frequency normal oscillations of the dissipative subsystem are damped. Evolution equations are constructed to describe the behaviour of the Hamiltonian subsystem over long time intervals.

1. SYSTEM DESCRIPTION. BASIC ASSUMPTIONS

Consider a mechanical system consisting of two interacting subsystems S_H and S_D.

When the interaction is removed, subsystem S_H becomes a Hamiltonian system with n degrees of freedom and subsystem S_D becomes a dissipative linear oscillatory system with m degrees of freedom. The characteristic period of oscillation in subsystem S_D and the characteristic damping time of these oscillations are comparable in magnitude and much smaller than the characteristic time of motions in subsystem S_H.

Below we call S_H the damped system and S_D the damper.

The equations of motion of the system $S_H + S_D$ can be written in Routhian form [1]

$$\mathbf{P}^\cdot = -\nabla_\mathbf{Q} R, \quad \mathbf{Q}^\cdot = \nabla_\mathbf{P} R, \quad (\nabla_\mathbf{v} R)^\cdot - \nabla_\mathbf{q} R = -\nabla_\mathbf{v}\Phi \tag{1.1}$$

Here $\mathbf{P} = (P_1, \ldots, P_n)$, $\mathbf{Q} = (Q_1, \ldots, Q_n)$ are canonical variables used to describe the motions in S_H and $\mathbf{q} = (q_1, \ldots, q_n)$ is the generalized coordinate vector of the damper with $\mathbf{v} = \dot{\mathbf{q}}$. Dots denote derivatives with respect to time t.

The Routhian function R in (1.1) is a combination of the Hamiltonian H subsystem S_H, the Lagrangian L of subsystem S_D, and a function K characterizing the interaction of the subsystems: $R = H + K - L$.

Given these assumptions the Lagrangian L and the dissipative function Φ of the damper can be written in the form

$$L(\mathbf{v}, \mathbf{q}, \varepsilon) = {}^1\!/\!_2[(\mathbf{v}, M\mathbf{v}) - \varepsilon^{-2}(\mathbf{q}, \Lambda\mathbf{q})], \quad \Phi(\mathbf{v}, \varepsilon) = {}^1\!/\!_2\varepsilon^{-1}(\mathbf{v}, D\mathbf{v}) \tag{1.2}$$
$$\varepsilon = T_D/T_H \ll 1$$

Here M, Λ and D are positive-definite symmetric matrices with constant coefficients, and T_D and T_H are characteristic times of processes in S_D and S_H.

We take

$$K(\mathbf{P}, \mathbf{Q}, \mathbf{v}, \mathbf{q}) = (\mathbf{u}, \mathbf{q}) + {}^1\!/\!_2(\mathbf{v}, \Gamma\mathbf{q}) + K_2(\mathbf{P}, \mathbf{Q}, \mathbf{q})$$

to be the interaction function with $\mathbf{u} = (u_1(\mathbf{P}, \mathbf{Q}), \ldots, u_m(\mathbf{P}, \mathbf{Q})) = \nabla_\mathbf{q} K(\mathbf{P}, \mathbf{Q}, 0, 0)$, Γ is an anti-symmetric matrix whose elements are functions of $\mathbf{P}$, $\mathbf{Q}$, and the function $K_2(\mathbf{P}, \mathbf{Q}, \mathbf{q}) = O(q^2)$, $q = |\mathbf{q}| = (q_1^2 + \ldots q_m^2)^{1/2}$.

With this choice of K the system $S_H + S_D$ is a finite-dimensional model of systems encountered in studies of the motion of a deformable solid about its centre of mass (see below, Section 6.)

We take the function H to be bounded in $R^n \times R^n$ together with derivatives of up to third order

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inclusive. The function $K(\mathbf{P}, \mathbf{Q}, \mathbf{v}, \mathbf{q})$ is assumed to be bounded together with its first and second derivatives in $R^n \times R^n \times \mathcal{B}_\Delta^m \times \mathcal{B}_\Delta^m$ (where $\mathcal{B}_\Delta^m$ is a sphere of radius Δ in R^m with centre at O).

2. STEADY MOTION

When studying the dynamics of $S_H + S_D$ over time intervals comparable to or substantially greater than T_H it is desirable to consider the motion of the damper to be forced and to describe it by relations of the form

$$\mathbf{v} = \mathbf{v}_*(\mathbf{P}, \mathbf{Q}, \varepsilon), \quad \mathbf{q} = \mathbf{q}_*(\mathbf{P}, \mathbf{Q}, \varepsilon) \tag{2.1}$$

Substituting (2.1) into the equations for $\dot{\mathbf{P}}, \dot{\mathbf{Q}}$ in (1.1) we obtain a closed system of equations describing the behaviour of subsystem S_H after the normal oscillations of the damper have decayed away.

Various modifications of these equations for the steady motions of specific systems were constructed in [2–4]. There have been attempts [5, 6] to give a vigorous justification for using such equations to describe the regular components of the motion by boundary function theory methods [7].

Relations (2.1) define a hypersurface Σ, $\dim \Sigma = 2m$ in the phase space of system $S_H + S_D$. If this hypersurface is invariant with respect to the phase flow of the system, it is called an integral manifold (IM) [8, 9].

Theorem. For sufficiently small values of the parameter ε system (1.1) possesses an IM Σ described by relations of the form (2.1). On the manifold Σ system (1.1) is equivalent to the system

$$\mathbf{P}^\cdot = -\nabla_{\mathbf{Q}} H - \nabla_{\mathbf{Q}} K(\mathbf{P}, \mathbf{Q}, \mathbf{v}_*(\mathbf{v}, \mathbf{q}, \varepsilon), \mathbf{q}_*(\mathbf{v}, \mathbf{q}, \varepsilon)) \tag{2.2}$$

$$\mathbf{Q}^\cdot = -\nabla_{\mathbf{P}} H + \nabla_{\mathbf{P}} K(\mathbf{P}, \mathbf{Q}, \mathbf{v}_*(\mathbf{v}, \mathbf{q}, \varepsilon), \mathbf{q}_*(\mathbf{v}, \mathbf{q}, \varepsilon))$$

The functions $\mathbf{v}_*(\mathbf{v}, \mathbf{q}, \varepsilon)$, $\mathbf{q}_*(\mathbf{v}, \mathbf{q}, \varepsilon)$ satisfy the inequalities

$$|\mathbf{v}_*(\mathbf{v}, \mathbf{q}, \varepsilon)| \leqslant \varepsilon^2 C_1, \quad |\mathbf{q}_*(\mathbf{v}, \mathbf{q}, \varepsilon)| \leqslant \varepsilon^2 C_1, \quad C_1 = \text{const} > 0$$

Proof. In Eqs (1.1) we replace the variables $\mathbf{v}$, $\mathbf{q}$ by the variables $\boldsymbol{\upsilon}$, $\boldsymbol{\xi}$ using the substitutions

$$\boldsymbol{\upsilon} = \varepsilon^{-1}\mathbf{v}, \quad \boldsymbol{\xi} = \varepsilon^{-2}\mathbf{q} + \Lambda^{-1}\mathbf{u}$$

In the new variables the motion is described by the singularly perturbed system of equations

$$\mathbf{P}^\cdot = -\nabla_{\mathbf{Q}} H - \nabla_{\mathbf{Q}} K(\mathbf{P}, \mathbf{Q}, \varepsilon\boldsymbol{\upsilon}, \varepsilon^2(\boldsymbol{\xi} - \Lambda^{-1}\mathbf{u})); \tag{2.3}$$

$$\mathbf{Q}' = \nabla_{\mathbf{P}} H + \nabla_{\mathbf{P}} K(\mathbf{P}, \mathbf{Q}, \varepsilon\boldsymbol{\upsilon}, \varepsilon^2(\boldsymbol{\xi} - \Lambda^{-1}\mathbf{u}))$$

$$\varepsilon = \left\| \begin{matrix} \boldsymbol{\upsilon}^\cdot \\ \boldsymbol{\xi}^\cdot \end{matrix} \right\| = \Xi \left\| \begin{matrix} \boldsymbol{\upsilon} \\ \boldsymbol{\xi} \end{matrix} \right\| + \varepsilon \left\| \begin{matrix} X_\upsilon(\mathbf{P}, \mathbf{Q}, \boldsymbol{\upsilon}, \boldsymbol{\xi}, \varepsilon) \\ X_\xi(\mathbf{P}, \mathbf{Q}, \boldsymbol{\upsilon}, \boldsymbol{\xi}, \varepsilon) \end{matrix} \right\|$$

Here

$$\Xi = \left\| \begin{matrix} -M^{-1}D & -M^{-1}\Lambda \\ E & 0 \end{matrix} \right\|$$

$$X_\upsilon = \Gamma(\mathbf{P}, \mathbf{Q})\boldsymbol{\upsilon} + O(\varepsilon^2), \quad X_\xi = \Lambda^{-1}\{\mathbf{u}, H\} + O(\varepsilon^2)$$

$$\{\mathbf{u}, H\} = (\{u_1, H\}, \ldots, \{u_m, H\})$$

$\{\cdot, \cdot\}$ are Poisson brackets for the subsystem S_H, and E is the unit $(m \times m)$ matrix.

System (2.3) satisfies the conditions of the existence theorem for IM in singularly perturbed systems of general form [8, pp. 265–271]. The proof of this theorem consists of constructing a special contraction mapping on the set of functions specifying hypersurfaces in phase space. For system (2.3) the construction of the mapping is much simpler. We intend to use this mapping to derive approximate equations for the steady motion, and hence describe it in detail.

Let $\mathcal{M}(s, S)$ be the set of pairs of vector functions of dimension m $(V(\mathbf{P}, \mathbf{Q}), Z(\mathbf{P}, \mathbf{Q}))$ satisfying the conditions

$$|V(\mathbf{P}', \mathbf{Q}') - V(\mathbf{P}, \mathbf{Q})| \leq s(|\mathbf{P}' - \mathbf{P}| + |\mathbf{Q}' - \mathbf{Q}|)$$
$$|Z(\mathbf{P}', \mathbf{Q}') - Z(\mathbf{P}, \mathbf{Q})| \leq s(|\mathbf{P}' - \mathbf{P}| + |\mathbf{Q}' - \mathbf{Q}|)$$
$$\sup_{R^n \times R^n} \max\{|V(\mathbf{P},\mathbf{Q})|, |Z(\mathbf{P},\mathbf{Q})|\} \leq S$$

We consider mapping $\mathcal{F}_\varepsilon$: $\mathcal{M}(s, S) \to \mathcal{M}(s, S)$ under which the pair (V, Z) turns into the pair $(\tilde{V}, \tilde{Z})$, where

$$\left\| \begin{matrix} \tilde{V}(\mathbf{P},\mathbf{Q}) \\ \tilde{Z}(\mathbf{P},\mathbf{Q}) \end{matrix} \right\| = \varepsilon \int_{-\infty}^{0} \exp(-\Xi\tau) \left\| \begin{matrix} \tilde{X}_\upsilon(\varepsilon\tau, \mathbf{P}, \mathbf{Q}, \varepsilon | V, Z) \\ \tilde{X}_\xi(\varepsilon\tau, \mathbf{P}, \mathbf{Q}, \varepsilon | V, Z) \end{matrix} \right\| d\tau$$
$$\tilde{X}_\zeta(t, \mathbf{P}, \mathbf{Q}, \varepsilon | V, Z) = X_\zeta(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}, V(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}), Z(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}), \varepsilon), \quad \zeta = \boldsymbol{\upsilon}, \boldsymbol{\xi}$$

The functions $\tilde{\mathbf{P}} = \tilde{\mathbf{P}}(t, \mathbf{P}, \mathbf{Q}, \varepsilon \mid V, Z)$, $\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}(t, \mathbf{P}, \mathbf{Q}, \varepsilon \mid V, Z)$ are a solution of the Cauchy problem

$$\tilde{\mathbf{P}}(0, \mathbf{P}, \mathbf{Q}, \varepsilon | V, Z) = \mathbf{P}, \quad \tilde{\mathbf{Q}}(0, \mathbf{P}, \mathbf{Q}, \varepsilon | V, Z) = \mathbf{Q}$$

for the closed system of equations

$$\mathbf{P}^{\cdot} = -\nabla_{\mathbf{Q}} H - \nabla_{\mathbf{Q}} K(\mathbf{P}, \mathbf{Q}, \varepsilon V, \varepsilon^2(Z - \Lambda^{-1}\mathbf{u}))$$
$$\mathbf{Q}^{\cdot} = -\nabla_{\mathbf{P}} H + \nabla_{\mathbf{P}} K(\mathbf{P}, \mathbf{Q}, \varepsilon V, \varepsilon^2(Z - \Lambda^{-1}\mathbf{u}))$$

It can be shown that

$$\varepsilon \left\| \begin{matrix} V(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}})^{\cdot} \\ Z(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}})^{\cdot} \end{matrix} \right\| = \Xi \left\| \begin{matrix} V(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}) \\ Z(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}) \end{matrix} \right\| + \varepsilon \left\| \begin{matrix} \tilde{X}_\upsilon \\ \tilde{X}_\xi \end{matrix} \right\|$$

With the condition $\varepsilon < \varepsilon_0(s, S)$, for (V_a, Z_a), $(V_b, Z_b) \in \mathcal{M}(s, S)$ we have the relations

$$\sup_{R^n \times R^n} \max\{|\tilde{V}_\zeta|, |\tilde{Z}_\zeta|\} \leq \varepsilon C_2 \quad (\zeta = a, b)$$
$$\sup_{R^n \times R^n} \max\{|\tilde{V}_a - \tilde{V}_b|, |\tilde{Z}_a - \tilde{Z}_b|\} \leq \tag{2.4}$$
$$\leq \varepsilon C_3 \sup_{R^n \times R^n} \max\{|V_a - V_b|, |Z_a - Z_b|\}$$
$$((\tilde{V}_a, \tilde{Z}_a) = \mathcal{F}_\varepsilon[(V_a, Z_a)], \quad (\tilde{V}_b, \tilde{Z}_b) = \mathcal{F}_\varepsilon[(V_b, Z_b)])$$

(where C_2 and C_3 are positive constants).

Because $\mathcal{F}_\varepsilon$ is a contraction mapping when $\varepsilon < \min\{\varepsilon_0, C_3^{-1}\}$, a pair $(V_*, Z_*) \in \mathcal{M}(s, S)$ exists such that $(V_*, Z_*) = \mathcal{F}_\varepsilon[(V_*, Z_*)]$. The relations

$$\boldsymbol{\upsilon} = V_*(\mathbf{P}, \mathbf{Q}, \varepsilon), \quad \boldsymbol{\xi} = Z_*(\mathbf{P}, \mathbf{Q}, \varepsilon) \tag{2.5}$$

define an IM of system (2.3). The functions V_* and Z_* depend on ε through its being the parameter of $\mathcal{F}_\varepsilon$.

The presence in system (2.3) of the IM (2.5) means the existence of the IM (2.1) for system (1.1). The theorem is proved.

The manifold Σ is an attractor. In some neighbourhood of Σ the change in the ratio of the actual and initial distances to the manifold Σ along an arbitrary solution of system (1.1) has an upper bound set by the function $C_5\exp(-C_4 t/\varepsilon)$, where C_4 and C_5 are positive constants [8, pp. 273–276]. Investigations of the asymptotic behaviour of the motion of subsystem S_H can therefore be restricted to analysing solutions lying on the manifold Σ.

3. PROPERTIES OF THE LOCAL DESCRIPTION OF STEADY MOTION

In the phase space of the damped system we select a close bounded domain $\mathfrak{D}$. A local IM $\Sigma_{\mathfrak{D}}$ will be a hypersurface in $\mathfrak{D}\times\mathfrak{B}_\Delta^m\times\mathfrak{B}_\Delta^m$ described by relations of the form (2.1) and satisfying the condition that any phase trajectory of system (1.1) starting on $\Sigma_{\mathfrak{D}}$ remains in $\Sigma_{\mathfrak{D}}$ as long as it does not leave $\mathfrak{D}\times\mathfrak{B}_\Delta^m\times\mathfrak{B}_\Delta^m$ [8].

Suppose that for systems $S_H^{(1)}+S_D^{(1)}$ and $S_H^{(2)}+S_D^{(2)}$the Routhian functions $R^{(1)}$ and $R^{(2)}$ are identical in $\mathfrak{D}\times\mathfrak{B}_\Delta^m\times\mathfrak{B}_\Delta^m$. Then $\Sigma_{\mathfrak{D}}^{(k)}=\Sigma^{(k)}\cap\mathfrak{D}\times\mathfrak{B}_\Delta^m\times\mathfrak{B}_\Delta^m$ ($k=1,2$) will be a local IM of both systems simultaneously.

We shall show that in the general case $\Sigma_{\mathfrak{D}}^{(1)}\not\equiv\Sigma_{\mathfrak{D}}^{(2)}$. Consider a domain $\mathfrak{D}$ and systems $S_H^{(1)}+S_D^{(1)}$, $S_H^{(2)}+S_D^{(2)}$ such that any phase trajectory of Eqs (1.1) with functions $R=R^{(1)}\equiv R^{(2)}$ belonging to $\Sigma_{\mathfrak{D}}^{(1)}$or $\Sigma_{\mathfrak{D}}^{(2)}$leaves $\mathfrak{D}\times\mathfrak{B}_\Delta^m\times\mathfrak{B}_\Delta^m$ as $t\to\infty$. In the domain $\mathfrak{D}$

$$\left\|\begin{matrix} V_*^{(1)}-V_*^{(2)} \\ Z_*^{(1)}-Z_*^{(2)} \end{matrix}\right\| = \varepsilon\int_{-\infty}^{T_{\mathfrak{D}}(\mathbf{P},\mathbf{Q})}\exp(-\Xi\tau)\left\|\begin{matrix} \tilde{X}_v^{(1)}-\tilde{X}_v^{(2)} \\ \tilde{X}_\xi^{(1)}-\tilde{X}_\xi^{(2)} \end{matrix}\right\|d\tau \tag{3.1}$$

Here $V_*^{(k)}$, $Z_*^{(k)}$ are functions defining $\Sigma_*^{(k)}$ for system $S_H^{(k)}+S_D^{(k)}$, and $T_{\mathfrak{D}}(\mathbf{P},\mathbf{Q})\leqslant 0$ is the time of intersection of the solution

$$\tilde{\mathbf{P}}(t,\mathbf{P},\mathbf{Q},\varepsilon|V_*^{(k)},Z_*^{(k)}),\quad \tilde{\mathbf{Q}}(t,\mathbf{P},\mathbf{Q},\varepsilon|V_*^{(k)},Z_*^{(k)})$$

with the boundary of the domain $\mathfrak{D}$ ($k=1,2$).

The value of the integral in (3.1) is governed by the behaviour of the functions $R^{(1)}$ and $R^{(2)}$ outside $\mathfrak{D}\times\mathfrak{B}_\Delta^m\times\mathfrak{B}_\Delta^m$ and is in general non-zero.

We will estimate the distance between the hypersurfaces $\Sigma_{\mathfrak{D}}^{(1)}$and $\Sigma_{\mathfrak{D}}^{(2)}$. We take a domain $\tilde{\mathfrak{D}}$ lying in $\mathfrak{D}$ together with some neighbourhood, and put

$$T_{\mathfrak{D}}(\tilde{\mathfrak{D}})=\max_{(\mathbf{P},\mathbf{Q})\in\partial\tilde{\mathfrak{D}}}T_{\mathfrak{D}}(\mathbf{P},\mathbf{Q})$$

Using the boundedness of the functions $X_v^{(k)},X_\xi^{(k)}$ ($k=1,2$) and relations (3.1), we find

$$\sup_{\mathfrak{D}}\max\{|V_*^{(1)}-V_*^{(2)}|,\ |Z_*^{(1)}-Z_*^{(2)}|\}\leqslant\varepsilon C_6\exp(-\kappa T_{\mathfrak{D}}(\tilde{\mathfrak{D}})/\varepsilon) \tag{3.2}$$

Here κ is the value of the real part of that root of the characteristic equation $\det|M\rho^2+D\rho+\Lambda|=0$ which is nearest to the imaginary axis, and C_6 is a positive constant.

It follows from (3.2) that the distance between $\Sigma_{\mathfrak{D}}^{(1)}$and $\Sigma_{\mathfrak{D}}^{(2)}$ in $\tilde{\mathfrak{D}}$ is of magnitude $O(\varepsilon^2\exp(-\kappa T_{\mathfrak{D}}(\tilde{\mathfrak{D}})/\varepsilon))$.

The non-uniqueness of the local IM should be taken into account when analysing the dynamics of specific systems S_H+S_D in situations when the application of the theorem proved in Section 2 is only possible by altering or redefining the functions $H(\mathbf{P},\mathbf{Q})$, $K(\mathbf{P},\mathbf{Q},\mathbf{v},\mathbf{q})$ in some domain of the system phase space.

4. APPROXIMATE EQUATIONS FOR STEADY MOTION OF THE SYSTEM

With an error of $O(\varepsilon^{k+1})$ the IM Σ_* of system (2.3) is described by relations $\mathbf{v}=V_k$, $\xi=Z_k$, where $(V_k,Z_k)=\mathcal{F}_\varepsilon^k[(0,0)]$. Indeed, we have the following chain of inequalities

$$\begin{aligned}
&\sup_{R^n\times R^n}\max\{|V_k-V_*|,\ |Z_k-Z_*|\}\leqslant\varepsilon C_3\sup\max\{|V_{k-1}-V_*|,\ |Z_{k-1}-Z_*|\}\leqslant\ldots\\
&\ldots\leqslant(\varepsilon C_3)^{(k-1)}\sup_{R^n\times R^n}\max\{|V_1-V_*|,\ |Z_1-Z_*|\}\leqslant\\
&\leqslant(\varepsilon C_3)^k\sup_{R^n\times R^n}\max\{|V_*|,\ |Z_*|\}\leqslant\varepsilon^{k+1}C_7,\quad C_7=C_2C_3^{k+1}
\end{aligned}$$

When $k=1$ we obtain

$$\begin{Vmatrix} V_1 \\ Z_1 \end{Vmatrix} = \varepsilon \int_{-\infty}^{0} \exp(-\Xi\tau) \begin{Vmatrix} \tilde{X}_\upsilon(\varepsilon\tau, \mathbf{P}, \mathbf{Q}, \varepsilon|0,0) \\ \tilde{X}_\xi(\varepsilon\tau, \mathbf{P}, \mathbf{Q}, \varepsilon|0,0) \end{Vmatrix} d\tau = -\varepsilon\Xi^{-1} \begin{Vmatrix} X_\upsilon(\mathbf{P}, \mathbf{Q}, 0, 0, \varepsilon) \\ X_\xi(\mathbf{P}, \mathbf{Q}, 0, 0, \varepsilon) \end{Vmatrix} +$$

$$+\varepsilon\Xi^{-1} \int_{-\infty}^{0} \exp(-\Xi\tau) \begin{Vmatrix} \dfrac{d}{d\tau}\tilde{X}_\upsilon(\varepsilon\tau, \mathbf{P}, \mathbf{Q}, \varepsilon|0,0) \\ \dfrac{d}{d\tau}\tilde{X}_\xi(\varepsilon\tau, \mathbf{P}, \mathbf{Q}, \varepsilon|0,0) \end{Vmatrix} d\tau =$$

$$= -\varepsilon\Xi^{-1} \begin{Vmatrix} 0 \\ \Lambda^{-1}\{\mathbf{u}, H\} \end{Vmatrix} + O(\varepsilon^2), \quad \Xi^{-1} = \begin{Vmatrix} 0 & E \\ -\Lambda^{-1}M & -\Lambda^{-1}D \end{Vmatrix}$$

Thus, with an error of $O(\varepsilon^2)$, the IM Σ_* is given by the relations

$$\mathbf{v} = -\varepsilon\Lambda^{-1}\{\mathbf{u}, H\}, \quad \boldsymbol{\xi} = \varepsilon\Lambda^{-1}D\Lambda^{-1}\{\mathbf{u}, H\}$$

From this it follows that for steady motions

$$\mathbf{v} = -\varepsilon^2\Lambda^{-1}\{\mathbf{u}, H\} \tag{4.1}$$

with an error of $O(\varepsilon^3)$, and

$$\mathbf{q} = -\varepsilon^2\Lambda^{-1}\mathbf{u} + \varepsilon^3\Lambda^{-1}D\Lambda^{-1}\{\mathbf{u}, H\} \tag{4.2}$$

with an error of $O(\varepsilon^4)$.

Substituting expressions (4.1), (4.2) into (2.2) we obtain a system of equations for the steady motion

$$\mathbf{P}^{\cdot} = -\nabla_{\mathbf{Q}}\mathcal{H} - \varepsilon^3 U_{\mathbf{Q}}\Lambda^{-1}D\Lambda^{-1}\{\mathbf{u}, H\}, \quad \mathbf{Q}^{\cdot} = \nabla_{\mathbf{P}}\mathcal{H} + \varepsilon^3 U_{\mathbf{P}}\Lambda^{-1}D\Lambda^{-1}\{\mathbf{u}, H\} \tag{4.3}$$

where

$$U_{\mathbf{P}} = \begin{Vmatrix} \dfrac{\partial u_1}{\partial P_1} \cdots \dfrac{\partial u_m}{\partial P_1} \\ \dots\dots\dots\dots \\ \dfrac{\partial u_1}{\partial P_n} \cdots \dfrac{\partial u_m}{\partial P_n} \end{Vmatrix}, \quad U_{\mathbf{Q}} = \begin{Vmatrix} \dfrac{\partial u_1}{\partial Q_1} \cdots \dfrac{\partial u_m}{\partial Q_1} \\ \dots\dots\dots\dots \\ \dfrac{\partial u_1}{\partial Q_n} \cdots \dfrac{\partial u_m}{\partial Q_n} \end{Vmatrix}$$

$$\mathcal{H}(\mathbf{P}, \mathbf{Q}, \varepsilon) = H(\mathbf{P}, \mathbf{Q}) + \varepsilon^2 H_2(\mathbf{P}, \mathbf{Q}), \quad H_2(\mathbf{P}, \mathbf{Q}) = -\tfrac{1}{2}(\mathbf{u}, \Lambda^{-1}\mathbf{u})$$

The nearly-Hamiltonian system of equations (4.3) describes the influence of the interaction with the damper on the dynamics of subsystem S_H to an accuracy of $O(\varepsilon)$ over a time interval ε^{-3}.

5. EVOLUTION OF STEADY MOTIONS IN AN INTEGRABLE SUBSYSTEM S_H

Suppose that $\mathbf{I} = (I_1, \dots, I_n)$, $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ are "action-angle" variables in S_H. In $\mathbf{I}$, $\boldsymbol{\varphi}$ variables the equations of steady motion have the form

$$\mathbf{I}^{\cdot} = -\varepsilon^2\nabla_{\varphi}H_2 - \varepsilon^3 U_{\varphi}\Lambda^{-1}D\Lambda^{-1}U_{\varphi}^T\boldsymbol{\omega}, \tag{5.1}$$

$$\boldsymbol{\varphi}^{\cdot} = \boldsymbol{\omega}(\mathbf{I}) + \varepsilon^2\nabla_{\mathbf{I}}H_2 + \varepsilon^3 U_{\mathbf{I}}\Lambda^{-1}D\Lambda^{-1}U_{\varphi}^T\boldsymbol{\omega}$$

Here $\boldsymbol{\omega}(\mathbf{I}) = \nabla_{\mathbf{I}}H(\mathbf{I})$ is the frequency vector of the subsystem S_H.

The variables of (5.1) separate: the $\mathbf{I}$ variables are slow ($\dot{\mathbf{I}} = O(\varepsilon^2)$) and the $\boldsymbol{\varphi}$ variables are fast ($\dot{\boldsymbol{\varphi}} = O(1)$).

We shall study the behaviour of the slow variables using an averaging method [10].

We restrict ourselves to the case when the Fourier series of the function $\mathbf{u}(\mathbf{I}, \boldsymbol{\varphi})$ with respect to $\boldsymbol{\varphi}$ contains a finite number of terms

$$\mathbf{u}(\mathbf{I},\varphi)=\sum_{\mathbf{k}\in Z^n,|\mathbf{k}|\leq N}\mathbf{u}_k(\mathbf{I})e^{i\langle\mathbf{k},\varphi\rangle},\quad \langle\mathbf{k},\varphi\rangle=k_1\varphi_1+\ldots+k_n\varphi_n$$

In system (5.1) we perform two consecutive averaging changes of variables

$$(\mathbf{I},\varphi)\xrightarrow{1}(\tilde{\mathbf{I}},\tilde{\varphi})\xrightarrow{2}(\tilde{\tilde{\mathbf{I}}},\tilde{\tilde{\varphi}})$$

The first change of variables removes the second-order terms in ε in the slow variable equations and is a canonical transformation with generating function

$$S(\tilde{\mathbf{I}},\varphi)=\langle\tilde{\mathbf{I}},\varphi\rangle-i\varepsilon^2\sum_{\mathbf{k}\in Z^n\setminus 0,\ |\mathbf{k}|\leq 2N}\frac{H_{2\mathbf{k}}(\mathbf{I})}{\langle\mathbf{k},\varphi\rangle}e^{i\langle\mathbf{k},\varphi\rangle}$$

where

$$H_{2\mathbf{k}}(\mathbf{I})=-\tfrac{1}{2}\sum_{\mathbf{k}'\in Z^n,\ |\mathbf{k}'|\leq N,\ |\mathbf{k}-\mathbf{k}'|\leq N}(\mathbf{u}_{\mathbf{k}'}(\mathbf{I}),\ \Lambda^{-1}\mathbf{u}_{\mathbf{k}-\mathbf{k}'}(\mathbf{I}))$$

The second change of variables removes terms of the third order in ε depending on φ from the slow variable equations.

In asymptotically small neighbourhoods of the resonance surfaces $\langle\omega(\mathbf{I}),\mathbf{k}\rangle=0$ ($\mathbf{k}\in Z^n$, $|\mathbf{k}|\leq 2N$) these changes of variables become meaningless. The properties of the solutions of system (5.1) at resonance must be investigated by the methods described in [11, Chapter III].

Far from the resonance surfaces the behaviour of the slow variables with accuracy $O(\varepsilon)$ in the time interval ε^{-3} are described by evolution equations (using the original notation for the averaged variables)

$$\mathbf{I}^{\cdot}=-\nabla_\omega\Phi_{\text{eff}}(\omega(\mathbf{I}),\mathbf{I})\tag{5.2}$$

where

$$\Phi_{\text{eff}}(\omega,\mathbf{I})=\varepsilon^3/2\langle\omega,D_{\text{eff}}\omega\rangle$$

$$D_{\text{eff}}=\langle\langle U_\varphi\Lambda^{-1}D\Lambda^{-1}U_\varphi^T\rangle\rangle=\sum_{\mathbf{k}\in Z^n,\ |\mathbf{k}|\leq N}(\mathbf{u}_\mathbf{k},\Lambda^{-1}D\Lambda^{-1}\mathbf{u}_{-\mathbf{k}})\mathbf{k}^T\mathbf{k}$$

$$\langle\langle\cdot\rangle\rangle=\frac{1}{(2\pi)^n}\int_0^{2\pi}\ldots\int_0^{2\pi}(\cdot)d\varphi_1\ldots d\varphi_n$$

The quadratic form $\Phi_{\text{eff}}(\omega,\mathbf{I})$ in (5.2) is an analogue of the function $\Phi(\mathbf{v},\epsilon)$ in (1.1) and describes the dissipation of energy in steady motion

$$\langle\langle\Phi(\mathbf{v}_*(\mathbf{I},\varphi,\varepsilon),\varepsilon)\rangle\rangle=\Phi_{\text{eff}}(\omega(\mathbf{I}),\mathbf{I})+O(\varepsilon^4)$$

6. THE $S_H + S_D$ SYSTEM AS A MODEL OF A DEFORMABLE SOLID PERFORMING TRANSLATIONAL-ROTATIONAL MOTION

In many investigations, for example, when studying the dynamics of large space structures or the tidal evolution of planetary rotation [4, 12, 13], the question arises of the translational–rotational motion of a deformable body in a potential field.

The motion of a deformable body with respect to its centre of mass consists of the rotation of the body as a whole and the elastic displacements s of its individual elements. The dissipation of mechanical energy during relative displacements leads to the damping of high-frequency normal oscillations and influences the motion of the body as a whole.

As a rule, the decay time of the natural oscillations is considerably less than the characteristic time of the motion of the body as a whole. Hence steady motion is fundamental for a deformable body.

We say that the system $S_H + S_D$ is an Nth order model if the subsystem S_H describes the motion of the body as a whole taking no account of deformation, while the subsystem S_D describes the deformation of the body on the basis of a finite-dimensional approximation of the deformation field **s**, using forms of free oscillation corresponding to the N lowest frequencies of the body.

As $N \to \infty$ the right-hand sides of equations for steady motion for models of corresponding order form a rapidly converging functional series. This enables us to consider low-order models using a qualitative analysis of the influence of deformations on the motion of specific objects.

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